

Multivariate Vandermonde matrices with separated nodes on the unit circle are stable

Stefan Kunis*[†] Dominik Nagel* Anna Strotmann*

March 15, 2021

We prove explicit lower bounds for the smallest singular value and upper bounds for the condition number of rectangular, multivariate Vandermonde matrices with scattered nodes on the complex unit circle. Analogously to the Shannon-Nyquist criterion, the nodes are assumed to be separated by a constant divided by the used polynomial degree. If this constant grows linearly with the spatial dimension, the condition number is uniformly bounded. If it grows only logarithmically with the spatial dimension, the condition number grows slightly stronger than exponentially with the spatial dimension. Both results are quasi optimal and improve over all previously known results of such type.

Key words and phrases: Vandermonde matrix, well-separated nodes, condition number, restricted Fourier matrices, frequency analysis, super resolution.

2010 AMS Mathematics Subject Classification : 15A18, 65T40, 42A15.

1 Introduction

The condition number of rectangular Vandermonde matrices with nodes on the complex unit circle became important for the stability analysis of subspace methods like the Matrix Pencil method [8], ESPRIT [19] and MUSIC [20], see also [21]. A deterministic performance analysis is provided in [1, 15, 17, 6] and relies on the smallest and the largest singular values of such Vandermonde matrices.

In the univariate case, the condition number and the extremal singular values were studied intensively during the last years. If nodes are on the unit circle and well-separated, tight upper bounds for the largest and lower bounds for the smallest singular value are proven in [17, 2, 7] by means of extremal functions. Furthermore, the situation in which nodes build clusters is investigated in [3, 14, 4, 12, 7, 11]. In the multivariate case, only few results are available: The matrix in question has full rank if the normalized node separation scales with the square root of the spatial dimension [9, 18] or with the logarithm of the spatial dimension [10], respectively. Some quantitative results are available in [13, 12] under a linear scaling in the spatial dimension.

*Osnabrück University, Institute of Mathematics {skunis, dnagel, astrotmann}@uos.de

[†]Osnabrück University, Research Center of Cellular Nanoanalytics

In this paper, we present new bounds for the extreme singular values of multivariate rectangular Vandermonde matrices with well-separated nodes on the complex unit circle. The main result is a lower bound for the smallest singular value under a separation condition that is logarithmically scaling with the dimension. This makes the result in [10] quantitative, the bound itself decays slightly stronger than exponential in the spatial dimension, see Theorem 3.5. A second result refines [13] and provides a dimension independent lower bound under a linearly scaling assumption on the separation, see Theorem 4.2. Table 2.1 anticipates the results and simplifies the comparison to already existing ones.

2 Preliminaries and previous results

Throughout the paper, $d \in \mathbb{N}$ always denotes the dimension and $\mathbb{T} := \mathbb{R}/\mathbb{Z} = [0, 1)$ the torus parametrizing the complex unit circle $\{z \in \mathbb{C} \mid |z| = 1\} = \{e^{2\pi i t} \in \mathbb{C} \mid t \in \mathbb{T}\}$. The *wrap-around distance* between two nodes $\mathbf{t}, \mathbf{t}' \in \mathbb{T}^d$ is defined by

$$|\mathbf{t} - \mathbf{t}'|_{\mathbb{T}^d} := \min_{\mathbf{r} \in \mathbb{Z}^d} \|\mathbf{t} - \mathbf{t}' + \mathbf{r}\|_{\infty}.$$

We note that this distance is the largest wrap-around distance in the coordinate directions. The *minimal separation distance* of a node set $\Omega = \{\mathbf{t}_1, \dots, \mathbf{t}_M\} \subset \mathbb{T}^d$ of cardinality $M \in \mathbb{N}$ is given by

$$q := \min_{\mathbf{t} \neq \mathbf{t}' \in \Omega} |\mathbf{t} - \mathbf{t}'|_{\mathbb{T}^d}.$$

For some parameter $N \in \mathbb{N}$, we call the node set *well-separated* if the minimal separation distance q fulfills $qN > 1$, see Figure 2.1 for an illustration.

Now set $\mathbf{z}_j := (e^{2\pi i \mathbf{t}_j}, \dots, e^{2\pi i \mathbf{t}_j})^\top \in \mathbb{C}^d$, $j = 1, \dots, M$, and let $\boldsymbol{\nu} := (\nu_1, \dots, \nu_d)^\top \in \mathbb{N}^d$ be a multi-index. We are interested in the multivariate, rectangular Vandermonde matrix

$$\mathbf{A} := \mathbf{A}_N(\Omega) := (\mathbf{z}_j^{\boldsymbol{\nu}})_{\substack{j=1, \dots, M \\ \boldsymbol{\nu} \in \mathbb{N}^d, \|\boldsymbol{\nu}\|_{\infty} < N}} \in \mathbb{C}^{M \times N^d}, \quad \mathbf{z}_j^{\boldsymbol{\nu}} := (\mathbf{z}_j)_1^{\nu_1} \cdots (\mathbf{z}_j)_d^{\nu_d}. \quad (2.1)$$

and its condition number $\text{cond}(\mathbf{A}) := \sigma_{\max}(\mathbf{A})/\sigma_{\min}(\mathbf{A})$ where $\sigma_{\max}(\mathbf{A})$ and $\sigma_{\min}(\mathbf{A})$ are the respective largest- and smallest singular values. A necessary condition for $\sigma_{\min}(\mathbf{A}) > 0$ and hence having a finite condition number is that the nodes are distinct (in general, this condition is sufficient if and only if $d = 1$). Furthermore, the continuity of the smallest singular value with respect to the entries in \mathbf{A} leads to $\lim_{q \rightarrow 0} \sigma_{\min}(\mathbf{A}) = 0$. Similarly, one obtains $\lim_{N \rightarrow \infty} \text{cond}(\mathbf{A}) = 1$. We continue with the collection of some known and easy to prove results in order to give a short overview.

Theorem 2.1. *With the above notation, we have the following results:*

i) *Without any further conditions, we have*

$$\sigma_{\min}(\mathbf{A}) \leq N^{\frac{d}{2}} \leq \sigma_{\max}(\mathbf{A})$$

with equality if and only if for each pair of distinct nodes $\mathbf{t}, \mathbf{t}' \in \Omega$ there exists a component $1 \leq s \leq d$ such that $N(\mathbf{t} - \mathbf{t}')_s \in \mathbb{Z} \setminus \{0\}$.

ii) *If $qN > 1$, then*

$$\left(N - \frac{1}{q}\right)^{\frac{1}{2}} \leq \sigma_{\min}(\mathbf{A}) \text{ for } d = 1 \text{ and } \sigma_{\max}(\mathbf{A}) \leq \left(N + \frac{1}{q}\right)^{\frac{d}{2}} \text{ for } d \geq 1.$$

iii) Finally, if $\Omega = \frac{1}{M}\{0, \dots, M-1\}^d$ is a set of equispaced nodes with $q = \frac{1}{M}$ and $qN \geq 1$, then

$$\left(N - \frac{1}{q}\right)^{\frac{d}{2}} \leq N^{\frac{d}{2}} \left(\frac{\lfloor Nq \rfloor}{Nq}\right)^{\frac{d}{2}} = \sigma_{\min}(\mathbf{A}) \leq \sigma_{\max}(\mathbf{A}) = N^{\frac{d}{2}} \left(\frac{\lceil Nq \rceil}{Nq}\right)^{\frac{d}{2}} \leq \left(N + \frac{1}{q}\right)^{\frac{d}{2}}$$

and the upper and lower bounds are tight for $q \searrow r/N$ and $q \nearrow r/N$, $r \in \mathbb{N}$, respectively.

Proof. The inequalities in the first result follow from the diagonal entries of $\mathbf{A}\mathbf{A}^*$ being N^d . Equality is equivalent to $\mathbf{A}\mathbf{A}^* = N^d \mathbf{I}_M$ (\mathbf{I}_M denotes the identity matrix of size M) and direct computation shows that these off diagonal are all zero if and only for any distinct pair of nodes \mathbf{t}, \mathbf{t}' there is at least one component of $\mathbf{t} - \mathbf{t}'$ being in $\mathbb{Z}/N \setminus \{0\}$. See Figure 2.1 for two examples and [5] for the result when $d = 1$.

The second result is due to [17, 2] for $d = 1$ and uses extremal minorant and majorant functions, respectively. A tensor product majorant can be used similarly to [16, Appendix A] to prove the upper bound when $d > 1$. The third result can be found in [13, Cor. 4.11]. ■

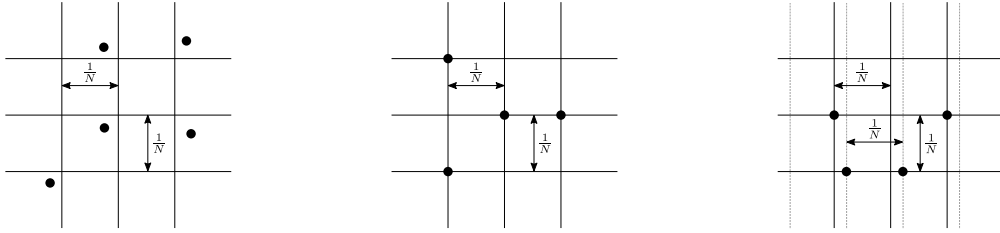


Figure 2.1: Left: Well-separated node set in \mathbb{T}^2 . Middle and right: Examples of node sets with perfectly conditioned Vandermonde matrix for $d = 2$; left: nodes on a grid with width $1/N$; right: each two distinct nodes have one coordinate direction with separation k/N , $0 \neq k \in \mathbb{Z}$.

Theorem 2.1 ii) and iii) show that the behaviour of the largest singular value with respect to the separation distance is completely understood for $qN > 1$. The very same is true for the smallest singular value only in the univariate situation $d = 1$. The multivariate situation is much more involved. Table 2.1 anticipates the result and simplifies the classification of the results and the comparison.

	$q(N-1) \geq$	$\sigma_{\min}(\mathbf{A})(N-1)^{-\frac{d}{2}} \geq$
Theorem 4.2 using [13, Cor. 4.7]	$4d$	0.9
[12, Ex. 4.4] and Theorem 4.1	$6d$	$\frac{1}{3}d^{-d/4}$
[18, Cor. 3.3] using [9]	\sqrt{d}	is positive
Remark 3.6 with $r = 1$	$1.01\sqrt{d} + 2$	$\frac{1}{2}d^{-d/4}$
[10, Cor. 2.10]	$3 + 2 \log d$	is positive
Theorem 3.5	$4.5 + 2.6 \log d$	$5.6^{-d} (1 + \log d)^{-d/4}$

Table 2.1: Bounds for the smallest singular value of multivariate Vandermonde matrices with well-separated nodes showing the dependency on the spatial dimension d .

3 Main result via a localizing function

The following function was already used in [10] to ensure the full rank of \mathbf{A} under the condition that the nodes are separated by $c_d/(N-1)$ where the constant $c_d > 1$ is logarithmically dependent on the spatial dimension d .

Definition 3.1 ([10]). *Let $d \in \mathbb{N}$ the dimension, $r \in \mathbb{N}$ and $p := 2r$. Furthermore, let $b, h > 0$ and $\varphi: \mathbb{R} \rightarrow \mathbb{R}$,*

$$\varphi(x) := \begin{cases} \left(1 - \left(\frac{2x}{h}\right)^2\right)^r, & |x| < \frac{h}{2}, \\ 0, & \text{else.} \end{cases}$$

We define the function $\psi: \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\psi := \left((2\pi b)^p - (-1)^r \sum_{s=1}^d \frac{\partial^p}{\partial x_s^p} \right) \bigotimes_{l=1}^d \varphi * \varphi. \quad (3.1)$$

Lemma 3.2 ([10, Lemma. 2.1]). *The function ψ from Definition 3.1 has the following properties:*

i) *Its Fourier transform $\widehat{\psi}(y) = \int_{\mathbb{R}^d} \psi(x) e^{-2\pi i y x} dx$ is bounded and it holds*

$$\widehat{\psi}(y) \begin{cases} \geq 0, & \|y\|_p \leq b, \\ \leq 0, & \|y\|_p \geq b, \end{cases}$$

ii) *its support is given by $\text{supp}(\psi) = [-h, h]^d$,*

iii) *$\psi(0) > 0$, if $h > C_p \sqrt[p]{d}/b$ with $C_p \leq \frac{2p+3}{e\pi}$.*

While these properties were sufficient for proving $\sigma_{\min}(\mathbf{A}) > 0$, they do not allow to directly deduce a quantitative lower bound. More advanced properties of ψ are presented in the next lemma and utilized for the proof of an explicit lower bound on the smallest singular value in Lemma 3.4 and Theorem 3.5.

Lemma 3.3. *Let Γ denote the gamma function. For the function ψ from Definition 3.1, we have*

$$\psi(0) = \left(\frac{h\sqrt{\pi}(2r)!}{2\Gamma(2r + \frac{3}{2})} \right)^d (2\pi b)^{2r} - \left(\frac{h\sqrt{\pi}(2r)!}{2\Gamma(2r + \frac{3}{2})} \right)^{d-1} \frac{d4^{2r}(r!)^2}{(2r+1)h^{2r-1}}$$

and

$$\max_{v \in \mathbb{R}^d} \widehat{\psi}(v) = \widehat{\psi}(0) = (2\pi b)^p \left(\frac{h\sqrt{\pi}r!}{2\Gamma(r + \frac{3}{2})} \right)^{2d}.$$

Proof. The proof [10, Lemma 2.1] already showed

$$\psi(0) = (\varphi * \varphi)^{d-1}(0) \left((2\pi b)^p \frac{h\sqrt{\pi}p!}{2\Gamma(p + \frac{3}{2})} - \frac{d4^p(r!)^2}{(p+1)h^{p-1}} \right).$$

Together with

$$\varphi * \varphi(0) = \frac{h}{2} \int_{-1}^1 (1-y^2)^p dy = \frac{h\sqrt{\pi}p!}{2\Gamma(p + \frac{3}{2})},$$

this yields the first result. The second claim follows from

$$\widehat{\psi}(v) = \left((2\pi b)^p - \sum_{s=1}^d (2\pi v_s)^p \right) \prod_{l=1}^d (\widehat{\varphi}(v_l))^2$$

and

$$|\widehat{\varphi}(v)| \leq \int_{\mathbb{R}} |\varphi(x)| dx = \widehat{\varphi}(0) = \frac{h}{2} \int_{-1}^1 (1-x^2)^r dx = \frac{h\sqrt{\pi}r!}{2\Gamma(r+\frac{3}{2})}.$$

■

Lemma 3.4. *For the function ψ from Definition 3.1, we have*

$$\frac{\psi(0)}{\widehat{\psi}(0)} > \left(\sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{r}}{h} \right)^d \left(1 - d \frac{2e^2}{\sqrt{\pi}} \sqrt{r} \left(\frac{2r}{\pi e \cdot bh} \right)^{2r} \right).$$

Choosing $h = \frac{2p+3}{e\pi} \sqrt[p]{d} \cdot \frac{1}{b}$ further yields

$$\frac{\psi(0)}{\widehat{\psi}(0)} > \frac{1}{2} \left(\frac{4}{3} \cdot \frac{b}{\sqrt{p} \cdot \sqrt[p]{d}} \right)^d$$

and setting $p = 2 \lceil \log(d) \rceil$ finally leads to $h \leq \frac{4 \log(d)+7}{b\pi}$ and

$$\frac{\psi(0)}{\widehat{\psi}(0)} > \frac{1}{2} \left(\frac{4}{3\sqrt{2}e^2} \cdot \frac{b}{\sqrt{\log(d)+1}} \right)^d.$$

Proof. We start from Lemma 3.3 and calculate

$$\begin{aligned} \frac{\psi(0)}{\widehat{\psi}(0)} &= \frac{\left(\frac{h\sqrt{\pi}(2r)!}{2\Gamma(2r+\frac{3}{2})} \right)^d (2\pi b)^{2r} - \left(\frac{h\sqrt{\pi}(2r)!}{2\Gamma(2r+\frac{3}{2})} \right)^{d-1} \frac{d4^{2r}(r!)^2}{(2r+1)h^{2r-1}}}{(2\pi b)^{2r} \left(\frac{h\sqrt{\pi}r!}{2\Gamma(r+\frac{3}{2})} \right)^{2d}} \\ &= \left(\frac{2(2r)!\Gamma^2(r+\frac{3}{2})}{\Gamma(2r+\frac{3}{2})h\sqrt{\pi}(r!)^2} \right)^d \left(1 - \frac{d2^{2r+1}\Gamma(2r+\frac{3}{2})(r!)^2}{(2r)!\Gamma(2r+\frac{3}{2})\sqrt{\pi}(2r+1)\pi^{2r}b^{2r}} \right). \end{aligned} \quad (3.2)$$

The Gautschi-Wendel inequality gives

$$\sqrt{x-\frac{1}{2}} < \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x)} < \sqrt{x+\frac{1}{2}}. \quad (3.3)$$

for $x > 1/2$. In combination with the definition of the Gamma function, we obtain for the expression in the first bracket of (3.2)

$$\begin{aligned} \frac{2(2r)!\Gamma^2(r+\frac{3}{2})}{\Gamma(2r+\frac{3}{2})h\sqrt{\pi}(r!)^2} &= \frac{2}{h\sqrt{\pi}} \cdot \frac{\Gamma(2r+1)}{\Gamma(2r+\frac{3}{2})} \cdot \left(\frac{\Gamma(r+\frac{3}{2})}{\Gamma(r+1)} \right)^2 > \frac{2}{h\sqrt{\pi}} \cdot \frac{r+\frac{1}{2}}{\sqrt{2r+\frac{3}{2}}} \\ &= \frac{1}{h\sqrt{\pi}} \cdot \frac{2r+1}{\sqrt{2r+\frac{3}{2}}} > \frac{\sqrt{2r+\frac{1}{2}}}{h\sqrt{\pi}} > \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{r}}{h}. \end{aligned} \quad (3.4)$$

Again with (3.3) and Stirling's formula, the second summand in the right bracketed term of (3.2) can be simplified to

$$\begin{aligned} \frac{2^{2r+1} d \Gamma(2r + \frac{3}{2}) (r!)^2}{(2r)! h^{2r} \sqrt{\pi} (2r+1) \pi^{2r} b^{2r}} &< \frac{2d}{\sqrt{\pi}} \left(\frac{2}{\pi b h} \right)^{2r} \frac{\sqrt{2r + \frac{3}{2}}}{2r+1} (r!)^2 \\ &\leq \frac{2d}{\sqrt{\pi}} \left(\frac{2}{\pi b h} \right)^{2r} \frac{\sqrt{2r + \frac{3}{2}}}{2r+1} e^{2r} r^{2r+1} e^{-2r} \\ &\leq d \frac{2e^2}{\sqrt{\pi}} \sqrt{r} \left(\frac{2r}{\pi e \cdot b h} \right)^{2r}. \end{aligned} \quad (3.5)$$

The choice $h = \frac{2p+3}{e\pi} \sqrt[2r]{d} \cdot \frac{1}{b}$ leads to

$$1 - d \frac{2e^2}{\sqrt{\pi}} \sqrt{r} \left(\frac{2r}{\pi e \cdot b h} \right)^{2r} = 1 - d \frac{2e^2}{\sqrt{\pi}} \sqrt{r} \left(\frac{2r}{(4r+3) \sqrt[2r]{d}} \right)^{2r} = 1 - \frac{2e^2}{\sqrt{\pi}} \sqrt{r} \left(\frac{2r}{(4r+3)} \right)^{2r} \geq \frac{1}{2} \quad (3.6)$$

and

$$\sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{r}}{h} = \frac{\sqrt{2r\pi b e}}{(4r+3) \sqrt[2r]{d}}. \quad (3.7)$$

Finally, setting $r = \lceil \log(d) \rceil$ and combining (3.5) to (3.7), yields

$$\frac{\psi(0)}{\widehat{\psi}(0)} > \frac{1}{2} \left(\frac{4}{3} \cdot \frac{b}{\sqrt{2 \log(d) + 2} \cdot d^{\frac{1}{2 \log(d) + 2}}} \right)^d > \frac{1}{2} \left(\frac{4b}{3\sqrt{2} \cdot e^2 \sqrt{\log(d) + 1}} \right)^d. \quad \blacksquare$$

Theorem 3.5. *Let \mathbf{A} be the Vandermonde matrix from (2.1) with well-separated nodes that further satisfy*

$$q(N-1) \geq \frac{8 \log(d) + 14}{\pi}.$$

Then the smallest singular value of \mathbf{A} is bounded by

$$\sigma_{\min}(\mathbf{A}) \geq \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{3e^2} \cdot \frac{1}{\sqrt{\log(d) + 1}} \right)^{\frac{d}{2}} (N-1)^{\frac{d}{2}}.$$

Proof. We follow the proof of [10, Cor. 2.5]. Since we have

$$\mathbf{A} = \text{diag} \left(\mathbf{z}_j^{\left(\lceil \frac{N-1}{2} \rceil, \dots, \lceil \frac{N-1}{2} \rceil \right)} \right) \cdot \widetilde{\mathbf{A}} \quad \text{with} \quad \widetilde{\mathbf{A}} := \left(\mathbf{z}_j^{\nu} \right)_{\substack{j=1, \dots, M \\ \nu \in \left\{ -\lceil \frac{N-1}{2} \rceil, \dots, \lfloor \frac{N-1}{2} \rfloor \right\}}}$$

and that singular values are unitary invariant, it holds $\sigma_{\min}(\mathbf{A}) = \sigma_{\min}(\widetilde{\mathbf{A}})$. Now, we use the function ψ from Definition 3.1 with parameters $b = \frac{N-1}{2}$, $p = 2 \log(d)$ and $h = \frac{8 \log(d) + 14}{(N-1)\pi}$ to obtain the following estimate by using the properties of ψ and the Poisson summation formula. Furthermore, notice that the condition on the separation distance q says that we have a h -separated node set. For arbitrary $\mathbf{u} \in \mathbb{C}^M$, we have

$$\widehat{\psi}(0) \left\| \widetilde{\mathbf{A}}^* \mathbf{u} \right\|^2 = \max_{\mathbf{x} \in \mathbb{R}^d} \widehat{\psi}(\mathbf{x}) \sum_{\nu \in \left\{ -\lceil \frac{N-1}{2} \rceil, \dots, \lfloor \frac{N-1}{2} \rfloor \right\}} \left| \sum_{j=1}^M (\mathbf{u})_j e^{2\pi i \nu^* \mathbf{t}_j} \right|^2$$

$$\begin{aligned}
&\geq \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \widehat{\psi}(\boldsymbol{\nu}) \left| \sum_{j=1}^M (\mathbf{u})_j e^{2\pi i \boldsymbol{\nu}^* \mathbf{t}_j} \right|^2 \\
&= \sum_{j=1}^M \sum_{\ell=1}^M (\mathbf{u})_j \overline{(\mathbf{u})_\ell} \sum_{\mathbf{r} \in \mathbb{Z}^d} \psi(\mathbf{t}_j - \mathbf{t}_\ell + \mathbf{r}) = \psi(\mathbf{0}) \sum_{j=1}^M \left| (\mathbf{u})_j \right|^2 = \psi(\mathbf{0}) \|\mathbf{u}\|^2.
\end{aligned}$$

Using the variational characterization of the smallest singular value leads to

$$\sigma_{\min}(\mathbf{A})^2 = \min_{\mathbf{u} \in \mathbb{C}^M \setminus \{0\}} \frac{\|\tilde{\mathbf{A}}^* \mathbf{u}\|^2}{\|\mathbf{u}\|^2} \geq \frac{\psi(0)}{\widehat{\psi}(0)}.$$

Applying Lemma 3.4 with the substitution of b with $(N-1)/2$ yields the result. \blacksquare

Remark 3.6. Under the assumed scaling $qN = a \log d + b \nearrow s$, $s \in \mathbb{N}$, $a > 0$, $b \in \mathbb{R}$, Theorem 2.1 iii) yields the almost matching bound

$$N^{-d} \sigma_{\min}^2(\mathbf{A}) \leq c_1 \left(1 - \frac{1}{a \log d + b} \right)^d \leq \exp \left(-\frac{c_2 \cdot d}{\log d} \right)$$

for some absolute constants $c_1, c_2 > 0$.

Using (3.2) directly for special small values of r leads to explicit expressions for $\psi(0)/\widehat{\psi}(0)$. After fixing r , the parameter h is chosen such the term becomes maximal,

$$h \cdot b = \begin{cases} \frac{\sqrt{5}}{\sqrt{2\pi}} \sqrt{d+2}, & r = 1, \\ \frac{\sqrt{3}}{\pi} \left(\frac{7}{2}\right)^{\frac{1}{4}} (d+4)^{\frac{1}{4}}, & r = 2, \\ \frac{\sqrt{3 \cdot 143^{1/6}}}{2^{1/3\pi}} (d+6)^{\frac{1}{6}}, & r = 3 \end{cases}$$

and we finally obtain

$$\frac{b^{-d} \psi(0)}{\widehat{\psi}(0)} = \begin{cases} 2^{\frac{3d}{2}+1} 5^{-\frac{3d}{2}} (d+2)^{-\frac{d}{2}-1} (3\pi)^d, & r = 1, \\ 2^{\frac{5d}{4}+2} 3^{-\frac{d}{2}} 7^{-\frac{5d}{4}} (d+4)^{-\frac{d}{4}-1} (5\pi)^d, & r = 2, \\ 2^{\frac{7d}{3}+1} 3^{1-\frac{3d}{2}} 143^{-\frac{7d}{6}} (d+6)^{-\frac{d}{6}-1} (175\pi)^d, & r = 3. \end{cases}$$

Resulting bounds for the smallest singular of \mathbf{A} and the conditions on the separation q from the choice of h , can be obtained analogously to the proof of Theorem 3.5 and are given in Table 3.1 for the first dimensions.

		$q(N-1) \geq$			$\sigma_{\min}(\mathbf{A}_N)(N-1)^{-\frac{d}{2}} \geq$		
$d \backslash r$	1	2	3	1	2	3	
1	1.744	2.014	2.251	0.677	0.421	0.246	
2	2.256	2.361	2.454	0.711	0.494	0.335	
3	2.769	2.831	2.887	0.710	0.499	0.347	

Table 3.1: Explicit constant of the separation condition and resulting bounds for the smallest singular value.

4 Further results

The following result for the case of well-separated nodes was already given as a special case of a result for multivariate clustered node configurations.

Theorem 4.1 (Lower bound on the smallest singular value,[12, Ex. 4.4]). *Let $M, d, N \in \mathbb{N}$, $N > \max \{M, 2(d+2)^2\}$, and $\mathbf{A} \in \mathbb{C}^{M \times N^d}$ be a Vandermonde matrix as in (2.1) with separation distance satisfying*

$$qN > 6d,$$

then we have

$$\sigma_{\min}(\mathbf{A}) \geq \frac{N^{d/2}}{3d^{d/4}}.$$

Finally, we present a lower bound on the smallest singular value that is proven by means of fast decaying trigonometric kernel functions in combination with the Gershgorin circle theorem, see [13].

Theorem 4.2. *Let $M, d, N \in \mathbb{N}$, $d \geq 2$, N being even and $\mathbf{A} \in \mathbb{C}^{M \times N^d}$ be a Vandermonde matrix as in (2.1) with separation distance satisfying*

$$qN > 4d,$$

then we have

$$\sigma_{\min}(\mathbf{A}) > 0.9N^{d/2}.$$

Proof. With a slightly different normalization and an irrelevant shift, Corollary 4.7 in [13] gives

$$\sigma_{\min}(\mathbf{A}\mathbf{D}\mathbf{A}^*) \geq 1 - \left(\frac{2d}{Nq}\right)^{d+1}$$

for $Nq > 2d$. The involved diagonal matrix $\mathbf{D} \in \mathbb{R}^{N^d \times N^d}$ has positive diagonal entries and the Rayleigh-Ritz characterization of the smallest eigenvalue of Hermitian matrices allows for the estimate

$$\begin{aligned} \sigma_{\min}(\mathbf{A}\mathbf{D}\mathbf{A}^*) &= \min_{\mathbf{x} \in \mathbb{C}^{N^d}, \|\mathbf{x}\|=1} \left\| \mathbf{D}^{1/2} \mathbf{A}^* \mathbf{x} \right\|^2 \\ &\leq \min_{\mathbf{x} \in \mathbb{C}^{N^d}, \|\mathbf{x}\|=1} \|\mathbf{D}\| \|\mathbf{A}^* \mathbf{x}\|^2 = \|\mathbf{D}\| \sigma_{\min}(\mathbf{A}\mathbf{A}^*). \end{aligned}$$

The diagonal matrix is given by a sampled B-spline and satisfies

$$\|\mathbf{D}\| \leq \frac{1}{N^d (1 - 2\zeta(d+1)(2\pi)^{-d-1})^d},$$

see [13, Thm. 3.3 and the proof of Cor. 3.5], where ζ denotes the Riemann zeta function. We finally use the assumption $Nq > 4d$, Bernoulli's inequality, and monotony of Riemann's zeta function to estimate

$$\begin{aligned} \sigma_{\min}^2(\mathbf{A}) &\geq N^d \left(1 - 2\zeta(d+1)(2\pi)^{-d-1}\right)^d \left(1 - 2^{-d-1}\right) \\ &\geq N^d \left(1 - 2d\zeta(3)(2\pi)^{-d-1}\right) \left(1 - 2^{-d-1}\right) \\ &> 0.85N^d. \end{aligned}$$

■

We close by noting that the result of Theorem 4.2 cannot be achieved under considerably weaker assumptions, since for $qN \in o(d)$, Theorem 2.1 iii) yields

$$N^{-d} \sigma_{\min}^2(\mathbf{A}) \leq \left(1 - \frac{1}{o(d)}\right)^d \rightarrow 0$$

for $d \rightarrow \infty$.

Acknowledgements. Support by the German Research Foundation (DFG-GK1916, DFG-SFB944) and Volkswagen Foundation is gratefully acknowledged.

References

- [1] C. Aubel and H. Bölcskei. Deterministic performance analysis of subspace methods for cisoid parameter estimation. In *IEEE International Symposium on Information Theory, ISIT 2016, Barcelona, Spain, July 10-15, 2016*, pages 1551–1555. IEEE, 2016.
- [2] C. Aubel and H. Bölcskei. Vandermonde matrices with nodes in the unit disk and the large sieve. *Appl. Comput. Harmon. Anal.*, 47(1):53–86, 2019.
- [3] D. Batenkov, L. Demanet, G. Goldman, and Y. Yomdin. Conditioning of partial nonuniform Fourier matrices with clustered nodes. *SIAM J. Matrix Anal. Appl.*, 41(1):199–220, 2020.
- [4] D. Batenkov, B. Diederichs, G. Goldman, and Y. Yomdin. The spectral properties of Vandermonde matrices with clustered nodes. *Linear Algebra Appl.*, 609:37–72, 2021.
- [5] L. Berman and A. Feuer. On perfect conditioning of Vandermonde matrices on the unit circle. *Electron. J. Linear Algebra*, 16:157–161, 2007.
- [6] S. Chrétien and H. Tyagi. Multi-kernel unmixing and super-resolution using the modified matrix pencil method. *J. Fourier Anal. Appl.*, 26(1):Paper No. 18, 2020.
- [7] B. Diederichs. Well-Posedness of Sparse Frequency Estimation. *arXiv e-prints*, May 2019.
- [8] Y. Hua and T. K. Sarkar. Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise. *IEEE Trans. Acoust. Speech Signal Process.*, 38(5):814–824, 1990.
- [9] V. Komornik and P. Loreti. Semi-discrete Ingham-type inequalities. *Appl. Math. Optim.*, 55(2):203–218, 2007.
- [10] S. Kunis, H. M. Möller, T. Peter, and U. von der Ohe. Prony’s method under an almost sharp multivariate Ingham inequality. *J. Fourier Anal. Appl.*, 24(5):1306–1318, 2018.
- [11] S. Kunis and D. Nagel. On the condition number of Vandermonde matrices with pairs of nearly-colliding nodes. *Numer. Algorithms*, Jul. 2020.
- [12] S. Kunis and D. Nagel. On the smallest singular value of multivariate Vandermonde matrices with clustered nodes. *Linear Algebra Appl.*, 604:1–20, 2020.

- [13] S. Kunis and D. Potts. Stability results for scattered data interpolation by trigonometric polynomials. *SIAM J. Sci. Comput.*, 29(4):1403–1419, 2007.
- [14] W. Li and W. Liao. Stable super-resolution limit and smallest singular value of restricted Fourier matrices. *Appl. Comput. Harmon. Anal.*, 51:118–156, 2021.
- [15] W. Li, W. Liao, and A. Fannjiang. Super-resolution limit of the ESPRIT algorithm. *IEEE Trans. Inform. Theory*, 66(7):4593–4608, 2020.
- [16] W. Liao. MUSIC for multidimensional spectral estimation: stability and super-resolution. *IEEE Trans. Signal Process.*, 63(23):6395–6406, 2015.
- [17] A. Moitra. Super-resolution, extremal functions and the condition number of Vandermonde matrices. In *STOC’15—Proceedings of the 2015 ACM Symposium on Theory of Computing*, pages 821–830. ACM, New York, 2015.
- [18] D. Potts and M. Tasche. Parameter estimation for multivariate exponential sums. *Electron. Trans. Numer. Anal.*, 40:204–224, 2013.
- [19] R. Roy and T. Kailath. ESPRIT - estimation of signal parameters via rotational invariance techniques. *IEEE Trans. Acoust. Speech Signal Process.*, 37(7):984–995, 1989.
- [20] R. O. Schmidt. Multiple emitter location and signal parameter estimation. *IEEE Trans. Antennas and Propagation*, 34(3):276–280, 1986.
- [21] P. Stoica and R. Moses. *Spectral Analysis of Signals*. Pearson Prentice Hall, 2005.