On the stability of the hyperbolic cross discrete Fourier transform

Lutz Kämmerer[†] Stefan Kunis[†‡]

A straightforward discretisation of problems in high dimensions often leads to an exponential growth in the number of degrees of freedom. Sparse grid approximations allow for a severe decrease in the number of used Fourier coefficients to represent functions with bounded mixed derivatives and the fast Fourier transform (FFT) has been adapted to this thin discretisation. We show that this so called hyperbolic cross FFT suffers from an increase of its condition number for both increasing refinement and increasing spatial dimension.

Key words and phrases : trigonometric approximation, hyperbolic cross, sparse grid, fast Fourier transform

2010 AMS Mathematics Subject Classification : 65T40, 65T50, 42A15, 15A60

1 Introduction

A straightforward discretisation of problems in d spatial dimensions with 2^n grid points in each coordinate leads to an exponential growth 2^{dn} in the number of degrees of freedom. Even an efficient algorithm like the d-dimensional fast Fourier transform (FFT) uses $C2^{dn}dn$ floating point operations. This is labelled as the curse of dimensions and the use of sparsity has become a very popular tool in such situations. For moderately high dimensional problems the use of sparse grids and the approximation on hyperbolic crosses has led to problems of total size $C_d 2^n n^{d-1}$. Moreover, the approximation rate hardly deteriorates for functions in an appropriate scale of spaces of dominating mixed smoothness, see e.g. [14, 16, 11, 10, 13, 2, 12, 15]. The FFT has been adapted to this thin discretisation as hyperbolic cross fast Fourier transform (HCFFT), which uses $C_d 2^n n^d$ floating point operations, in [1, 9, 7], see also [5] for a recent generalisation to arbitrary spatial sampling nodes and [4] for the associated Matlab toolbox.

In this paper, we consider the numerical stability of the hyperbolic cross discrete Fourier transform, which of course limits the stability of a particular and potentially fast algorithm like the HCFFT. While the ordinary discrete Fourier transform is up to some constant a unitary transform and thus has condition number one, its hyperbolic cross version suffers

[†]Chemnitz University of Technology, Faculty of Mathematics, {kaemmerer,kunis}@mathematik.tu-chemnitz.de

[‡]Helmholtz Zentrum München, Institute for Biomathematics and Biometry

$n \diagdown d$	2	3	4	5	6	7	8	9	10
1	2.00	3.73	6.34	$9.90 \cdot 10^0$	$1.44 \cdot 10^1$	$1.99 \cdot 10^1$	$2.65 \cdot 10^1$	$3.40 \cdot 10^1$	$4.25 \cdot 10^1$
2	3.24	9.49	22.84	$4.71 \cdot 10^1$	$8.72 \cdot 10^1$	$1.49 \cdot 10^2$	$2.41 \cdot 10^2$	$3.72 \cdot 10^2$	$5.52 \cdot 10^2$
3	5.15	19.43	60.79	$1.58 \cdot 10^2$	$3.57 \cdot 10^2$	$7.34 \cdot 10^2$	$1.40 \cdot 10^3$	$2.51 \cdot 10^3$	$4.29 \cdot 10^3$
4	8.08	36.21	135.74	$4.26 \cdot 10^2$	$1.15 \cdot 10^3$	$2.78 \cdot 10^3$	$6.14 \cdot 10^3$	$1.26 \cdot 10^4$	$2.45 \cdot 10^4$
5	12.53	63.85	272.26	$1.01 \cdot 10^3$	$3.17 \cdot 10^3$	$8.82 \cdot 10^3$	$2.22 \cdot 10^4$	$5.16 \cdot 10^4$	$1.12 \cdot 10^5$
6	19.21	108.72	518.01	$2.17 \cdot 10^3$	$7.80 \cdot 10^3$	$2.46 \cdot 10^4$	$6.98 \cdot 10^4$	$1.81 \cdot 10^5$	$4.38 \cdot 10^5$

Table 1.1: Condition number of the hyperbolic cross discrete Fourier transform.

from an increase of its condition number for both increasing refinement n and increasing spatial dimension d. We illustrate this behaviour in Table 1.1, which shows that already for spatial dimension $d = 9$ and a refinement $n = 4$ the HCFFT looses four digits of accuracy for a worst case input. As a rule of thumb for fixed dimension, the condition number at least doubles whenever the refinement is increased by two.

The paper is organised as follows: After introducing the necessary notation and collecting basic facts about the hyperbolic cross and related sets, we discuss the interpolation of functions by trigonometric polynomials. By convenience, we use the term "ordinary Fourier matrix" for the full grid case and reserve the short hand "Fourier matrix" for the hyperbolic cross and sparse grid case. We start by the interpolation on a full grid which leads to a trigonometric polynomial of an appropriate multi-degree and give the well known formulation as discrete Fourier transform, i.e., as matrix vector product with the ordinary inverse Fourier matrix. Subsequently, we consider the interpolation on the sparse grid which leads to a trigonometric polynomial on the hyperbolic cross. Since the interpolation operator has a Boolean sum decomposition, the associated inverse Fourier matrix allows for two similar decompositions in ordinary inverse Fourier matrices as well. In particular, this yields upper bounds for the norm of these inverse Fourier matrices, cf. Lemmata 2.3 and 2.5. We proceed by computing the Fourier coefficients of the Lagrange interpolant, interpolating one at the origin and zero at all other sparse grid nodes in Section 2.3 - which yields lower bounds for the norm of these inverse Fourier matrices. The main results of this paper are estimates on the norms of the Fourier matrices and their inverses in Theorems 3.1 and 4.1 for fixed spatial dimension and fixed refinement, respectively. These results are refined in Lemmata 3.4 and 4.4 for $d = 2$ and $n = 1$, respectively. All theoretical results are illustrated by a couple of numerical experiments. Finally, we conclude our findings in Section 5.

2 Prerequisite

Throughout this paper let the spatial dimension $d \in \mathbb{N}$ and a refinement $n \in \mathbb{N}_0$ be given. We denote by $\mathbb{T}^d \cong [0,1)^d$ the d-dimensional torus and consider Fourier series $f : \mathbb{T}^d \to \mathbb{C}$, $f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}}$ with Fourier coefficients $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$. The space of trigonometric polynomials $\Pi_{\boldsymbol{j}}$, $\boldsymbol{j} \in \mathbb{N}_0^d$, consists of all such series with Fourier coefficients supported on $\hat{G}_{\boldsymbol{j}} = \times_{l=1}^d \hat{G}_{j_l}$, $\hat{G}_j = \mathbb{Z} \cap (-2^{j-1}, 2^{j-1}]$, i.e., $f : \mathbb{T}^d \to \mathbb{C}$,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \hat{G}_{\boldsymbol{j}}} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}}.$$

A well adapted spatial discretisation of trigonometric polynomials relies on the full spatial grid $G_{\boldsymbol{j}} = \times_{l=1}^d G_{j_l}$, $G_j = 2^{-j}(\mathbb{Z} \cap [0, 2^j))$. If all refinements are equally set to $j_l = n$, $l = 1, \ldots, d$,

this yields 2^{dn} degrees of freedom in frequency as well as in spatial domain.

2.1 Hyperbolic cross and sparse grid

For functions of appropriate smoothness, it is much more effective to restrict the frequency domain to the hyperbolic cross

$$H_n^d := \bigcup_{\substack{\|\boldsymbol{j}\|_1=n \\ \boldsymbol{j}\in\mathbb{N}_0^d}} \hat{G}_{\boldsymbol{j}} = \{\boldsymbol{k} \in \hat{G}_{\boldsymbol{j}} : \|\boldsymbol{j}\|_1 = n\} \qquad \subset \hat{G}_n \times \ldots \times \hat{G}_n \subset \mathbb{Z}^d, \tag{2.1}$$

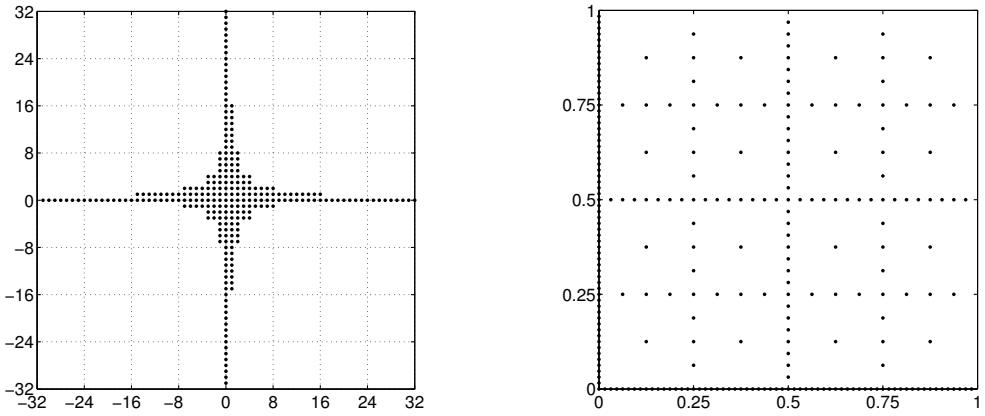
see Figure 2.1(a). This leads to the space Π_n^{hc} of trigonometric polynomials on the hyperbolic cross, i.e., $f : \mathbb{T}^d \to \mathbb{C}$,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in H_n^d} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}}.$$

Here, an appropriate spatial discretisation is given by the sparse grid

$$S_n^d := \bigcup_{\substack{\|\boldsymbol{j}\|_1=n \\ \boldsymbol{j}\in\mathbb{N}_0^d}} G_{\boldsymbol{j}} = \{\boldsymbol{x} \in G_{\boldsymbol{j}} : \|\boldsymbol{j}\|_1 = n\} \qquad \subset G_n \times \ldots \times G_n \subset \mathbb{T}^d, \tag{2.2}$$

see Figure 2.1(b). For notational convenience, we set $H_{-1}^d := S_{-1}^d := \hat{G}_{-1} := G_{-1} := \emptyset$. An



(a) Hyperbolic cross $H_6^2 \subset \mathbb{Z}^2$. (b) Sparse grid $S_6^2 \subset \mathbb{T}^2$.

Figure 2.1: Two dimensional hyperbolic cross and corresponding sparse grid.

immediate consequence of (2.1) is the partition

$$H_n^d = \bigcup_{s=0}^{n} H_{n-s}^{d-1} \times (\hat{G}_s \setminus \hat{G}_{s-1}). \tag{2.3}$$

We proceed by two lemmata on the cardinality of the hyperbolic cross and related sets.

Lemma 2.1. *For $d, n \in \mathbb{N}$, we have the cardinality estimates $\left|H_0^d\right| = \left|S_0^d\right| = 1$,*

$$|S_n^d| = |H_n^d| = \sum_{j=0}^{\min(n,d-1)} 2^{n-j} \binom{n}{j} \binom{d-1}{j} = \begin{cases} \frac{2^n n^{d-1}}{2^{d-1}(d-1)!} + \mathcal{O}(2^n n^{d-2}) & \text{for fixed } d \geq 2 \\ \frac{d^n}{n!} + \mathcal{O}(d^{n-1}) & \text{for fixed } n, \end{cases}$$

and

$$\left|H_n^d \setminus H_{n-1}^d\right| = \begin{cases} \displaystyle\sum_{j=0}^{d-1} 2^{n-j-1} \frac{n+j}{n-j} \binom{n-1}{j} \binom{d-1}{j}, & \text{for } n \geq d \\ \displaystyle\sum_{j=0}^{n-1} 2^{n-j-1} \frac{n+j}{n-j} \binom{n-1}{j} \binom{d-1}{j} + \binom{d-1}{n}, & \text{for } n < d. \end{cases}$$

In particular, this yields

i) $\left|H_n^d \setminus H_{n-1}^d\right| \geq \frac{2^n n^{d-1}}{2^d (d-1)!} - \mathcal{O}(2^n n^{d-2})$ *for fixed* $d \in \mathbb{N}$ *and* $n \geq d$*, and*

ii) $|H_0^2 \setminus H_{-1}^2| = 1$ *and* $|H_n^2 \setminus H_{n-1}^2| = (n+3)2^{n-2}$ *for* $n > 0$ *and* $d = 2$*.*

Proof. The cardinality estimate for the hyperbolic cross and the sparse grid is well known and can be found for example in [9]. Regarding the set differences, we only consider $n < d$ since the other case follows analogously. Due to $H_{n-1}^d \subset H_n^d$, we compute

$$\begin{aligned} |H_n^d| - |H_{n-1}^d| &= \sum_{j=0}^{n} 2^{n-j} \binom{n}{j} \binom{d-1}{j} - \sum_{j=0}^{n-1} 2^{n-1-j} \binom{n-1}{j} \binom{d-1}{j} \\ &= \sum_{j=0}^{n-1} 2^{n-1-j} \binom{d-1}{j} \left(2\binom{n}{j} - \binom{n-1}{j}\right) + \binom{d-1}{n} \\ &= \sum_{j=0}^{n-1} 2^{n-1-j} \binom{d-1}{j} \frac{n+j}{n-j} \binom{n-1}{j} + \binom{d-1}{n}. \end{aligned}$$

The asymptotic estimate i) can be seen from the summand $j = d-1$, ii) can be computed explicitly. ∎

Lemma 2.2. *Let* $d \in \mathbb{N}$*,* $n \in \mathbb{N}_0$*, and* $\boldsymbol{k} \in \mathbb{Z}^d$ *be given and set* $\ell(\boldsymbol{k}) := \min\left\{l \in \mathbb{N}_0 : \boldsymbol{k} \in H_l^d\right\}$*. Then we have*

$$\left|\{\boldsymbol{m} \in \mathbb{N}_0^d : \|\boldsymbol{m}\|_1 = n \text{ and } \boldsymbol{k} \in \hat{G}_{\boldsymbol{m}}\}\right| = \begin{cases} \binom{n-\ell(\boldsymbol{k})+d-1}{d-1} & \ell(\boldsymbol{k}) \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First note that each $\boldsymbol{k} \in \mathbb{Z}^d$ allows for exactly one $\boldsymbol{l} \in \mathbb{N}_0^d$ with $k_j \in \hat{G}_{l_j} \setminus \hat{G}_{l_j-1}$, $j = 1, \ldots, d$. Moreover, this multi-index fulfils $\|\boldsymbol{l}\|_1 = \ell(\boldsymbol{k})$ and thus $\boldsymbol{k} \in \hat{G}_{\boldsymbol{m}}$ if and only if $m_j = l_j + r_j$, $r_j \in \mathbb{N}_0$, $j = 1, \ldots, d$. In summary, we obtain

$$\left|\{\boldsymbol{m} \in \mathbb{N}_0^d : \|\boldsymbol{m}\|_1 = n \text{ and } \boldsymbol{k} \in \hat{G}_{\boldsymbol{m}}\}\right| = \left|\{\boldsymbol{r} \in \mathbb{N}_0^d : \|\boldsymbol{r}\|_1 = n - \ell(\boldsymbol{k})\}\right|,$$

from which the assertion follows by simple combinatorics. ∎

2.2 Operators and associated matrices

Interpolation operators typically take samples at certain sampling nodes and construct a function from a particular linear space which can be represented by its expansion coefficients. Hence, the linear map from sample values to expansion coefficients and vice versa is given by a matrix which is of interest. We start by reviewing the classical trigonometric interpolation on the full grid. For $d \in \mathbb{N}$, $\boldsymbol{j} \in \mathbb{N}_0^d$, and continuous functions $f \in C(\mathbb{T}^d)$, we define the interpolation operator $\mathcal{I}_{\boldsymbol{j}} : C(\mathbb{T}^d) \to \Pi_{\boldsymbol{j}}$ by the conditions

$$\mathcal{I}_{\boldsymbol{j}} f(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in G_{\boldsymbol{j}}.$$

Clearly, we have $\mathcal{I}_{\boldsymbol{j}} = \mathcal{I}_{j_1} \otimes \ldots \otimes \mathcal{I}_{j_d}$ and

$$\mathcal{I}_{\boldsymbol{j}} f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \hat{G}_{\boldsymbol{j}}} \hat{f}_{\boldsymbol{j},\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}}, \qquad \hat{f}_{\boldsymbol{j},\boldsymbol{k}} = \frac{1}{2^{\|\boldsymbol{j}\|_1}} \sum_{\boldsymbol{x} \in G_{\boldsymbol{j}}} f(\boldsymbol{x}) \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}}.$$

Moreover, the discrete Fourier coefficients coincide with the Fourier coefficients for trigonometric polynomials of multi-degree $2^{\boldsymbol{j}}$, i.e., $\hat{f}_{\boldsymbol{j},\boldsymbol{k}} = \hat{f}_{\boldsymbol{k}}$, $\boldsymbol{k} \in \hat{G}_{\boldsymbol{j}}$, for $f \in \Pi_{\boldsymbol{j}}$. In matrix vector notation, i.e.,

$$\hat{\boldsymbol{f}}_{\boldsymbol{j}} = (\hat{f}_{\boldsymbol{j},\boldsymbol{k}})_{\boldsymbol{k} \in \hat{G}_{\boldsymbol{j}}} \in \mathbb{C}^{|\hat{G}_{\boldsymbol{j}}|}, \qquad \boldsymbol{f} = (f_{\boldsymbol{x}})_{\boldsymbol{x} \in G_{\boldsymbol{j}}} = (f(\boldsymbol{x}))_{\boldsymbol{x} \in G_{\boldsymbol{j}}} \in \mathbb{C}^{|G_{\boldsymbol{j}}|},$$

we have

$$\hat{\boldsymbol{f}}_{\boldsymbol{j}} = \boldsymbol{F}_{\boldsymbol{j}}^{-1} \boldsymbol{f}, \qquad \boldsymbol{F}_{\boldsymbol{j}}^{-1} = \frac{1}{|\hat{G}_{\boldsymbol{j}}|} (\mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}})_{\boldsymbol{k} \in \hat{G}_{\boldsymbol{j}}, \boldsymbol{x} \in G_{\boldsymbol{j}}},$$

where

$$\boldsymbol{F}_{\boldsymbol{j}} := (\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}})_{\boldsymbol{x} \in G_{\boldsymbol{j}}, \boldsymbol{k} \in \hat{G}_{\boldsymbol{j}}} = \boldsymbol{F}_{j_1} \otimes \ldots \otimes \boldsymbol{F}_{j_d}$$

denotes the ordinary Fourier matrix.

Next, we turn to the trigonometric interpolation on the sparse grid, see the monograph [3] for an introduction. In what follows, the interpolation operator allows for a Boolean sum decomposition which is used for analysing the associated Fourier matrix. For $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, and continuous functions $f \in C(\mathbb{T}^d)$, we define the interpolation operator $\mathcal{L}_n^d : C(\mathbb{T}^d) \to \Pi_n^{\mathrm{hc}}$ by the conditions

$$\mathcal{L}_n^d f(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in S_n^d.$$

This time, we have the Boolean sum decomposition

$$\mathcal{L}_n^d = \bigoplus_{\substack{\boldsymbol{j} \in \mathbb{N}_0^d \\ \|\boldsymbol{j}\|_1 = n}} \mathcal{I}_{\boldsymbol{j}}$$

and

$$\mathcal{L}_n^d f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in H_n^d} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}}, \qquad \hat{\boldsymbol{f}} = (\boldsymbol{F}_n^d)^{-1} \boldsymbol{f},$$

for all trigonometric polynomials on the hyperbolic cross $f \in \Pi_n^{\mathrm{hc}}$, where

$$\boldsymbol{F}_n^d := (\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}})_{\boldsymbol{x} \in S_n^d, \boldsymbol{k} \in H_n^d}$$

denotes the Fourier matrix and we drop the superscript for $d = 1$. Moreover, the interpolation operator fulfils the well known relation

$$\mathcal{L}_n^d = \sum_{j=0}^{n} \mathcal{I}_{n-j} \otimes \mathcal{L}_j^{d-1} - \sum_{j=0}^{n-1} \mathcal{I}_{n-1-j} \otimes \mathcal{L}_j^{d-1} \tag{2.4}$$

which gives rise to the following result.

Lemma 2.3. *Let $d \in \mathbb{N}$, $d \geq 2$, and $n \in \mathbb{N}_0$, then the inverse Fourier matrices fulfil $\|\boldsymbol{F}_n^{-1}\|_2^2 = 2^{-n}$ and*

$$\|(\boldsymbol{F}_n^d)^{-1}\|_2 \leq \sum_{j=0}^{n} \|\boldsymbol{F}_{n-j}^{-1}\|_2 \|(\boldsymbol{F}_j^{d-1})^{-1}\|_2 + \sum_{j=0}^{n-1} \|\boldsymbol{F}_{n-1-j}^{-1}\|_2 \|(\boldsymbol{F}_j^{d-1})^{-1}\|_2.$$

Proof. Each individual summand $\mathcal{I}_{n-j} \otimes \mathcal{L}_j^{d-1}$ in (2.4) takes samples only from the set $G_{n-j} \times S_j^{d-1} \subset S_n^d$ as its input and produces a certain trigonometric polynomial with Fourier coefficients supported only on the set $\hat{G}_{n-j} \times H_j^{d-1} \subset H_n^d$. For subsequent use, let $X \subset S_n^d$, $Y \subset H_n^d$, the restriction matrix $\boldsymbol{P}_n^d(X) \in \mathbb{R}^{|X| \times |S_n^d|}$ and the extension matrix $\boldsymbol{Q}_n^d(Y) \in \mathbb{R}^{|H_n^d| \times |Y|}$,

$$(\boldsymbol{P}_n^d(X)\boldsymbol{f})_{\boldsymbol{x}} = f_{\boldsymbol{x}},\ \boldsymbol{x} \in X, \qquad (\boldsymbol{Q}_n^d(Y)\hat{\boldsymbol{f}})_{\boldsymbol{k}} = \begin{cases} \hat{f}_{\boldsymbol{k}} & \boldsymbol{k} \in Y, \\ 0 & \boldsymbol{k} \in H_n^d \setminus Y, \end{cases}$$

be given. Thus, the inverse Fourier matrix allows for the representation

$$\begin{aligned}(\boldsymbol{F}_n^d)^{-1} = &\sum_{j=0}^{n} \boldsymbol{Q}_n^d(\hat{G}_{n-j} \times H_j^{d-1}) \left(\boldsymbol{F}_{n-j}^{-1} \otimes (\boldsymbol{F}_j^{d-1})^{-1}\right) \boldsymbol{P}_n^d(G_{n-j} \times S_j^{d-1}) \\ &- \sum_{j=0}^{n-1} \boldsymbol{Q}_n^d(\hat{G}_{n-1-j} \times H_j^{d-1}) \left(\boldsymbol{F}_{n-1-j}^{-1} \otimes (\boldsymbol{F}_j^{d-1})^{-1}\right) \boldsymbol{P}_n^d(G_{n-1-j} \times S_j^{d-1}).\end{aligned}$$

Since the restriction and extension matrices have norms bounded by one, the triangle inequality yields the assertion. ∎

Example 2.4. *The decomposition of the previous Lemma, for $d = 2$ and $n = 1$, is*

$$\begin{aligned}(\boldsymbol{F}_1^2)^{-1} &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \boldsymbol{Q}_1^2(\hat{G}_1 \times \hat{G}_0) \left(\boldsymbol{F}_1^{-1} \otimes \boldsymbol{F}_0^{-1}\right) \boldsymbol{P}_1^2(G_1 \times G_0) \\ &+ \boldsymbol{Q}_1^2(\hat{G}_0 \times \hat{G}_1) \left(\boldsymbol{F}_0^{-1} \otimes \boldsymbol{F}_1^{-1}\right) \boldsymbol{P}_1^2(G_0 \times G_1) \\ &- \boldsymbol{Q}_1^2(\hat{G}_0 \times \hat{G}_0) \left(\boldsymbol{F}_0^{-1} \otimes \boldsymbol{F}_0^{-1}\right) \boldsymbol{P}_1^2(G_0 \times G_0) \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

which yields by the triangle inequality the norm estimate

$$1 = \|(\boldsymbol{F}_1^2)^{-1}\|_2 \leq \|(\boldsymbol{F}_1^1)^{-1}\|_2 + \|(\boldsymbol{F}_1^1)^{-1}\|_2 + \|(\boldsymbol{F}_0^1)^{-1}\|_2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 = 1 + \sqrt{2}.$$

Similar to the combination technique for sparse grids [8, 6], we have the following result for sums of interpolation operators of a specific level and their associated matrices.

Lemma 2.5. *For $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, let the matrices $\boldsymbol{\Sigma}_{n-l}^d \in \mathbb{C}^{|H_{n-l}^d| \times |S_{n-l}^d|}$, $l = 0, \ldots, \min(n, d-1)$ be given by*

$$\boldsymbol{\Sigma}_{n-l}^d = \sum_{\substack{\boldsymbol{j} \in \mathbb{N}_0^d \\ \|\boldsymbol{j}\|_1 = n-l}} \boldsymbol{Q}_{n-l}^d(\hat{G}_{\boldsymbol{j}}) \boldsymbol{F}_{\boldsymbol{j}}^{-1} \boldsymbol{P}_{n-l}^d(G_{\boldsymbol{j}}),$$

where the restriction matrices $\boldsymbol{P}_{n-l}^d$ and the extension matrices $\boldsymbol{Q}_{n-l}^d$ are given as in Lemma 2.3. Then, the inverse Fourier matrix allows for the decomposition

$$(\boldsymbol{F}_n^d)^{-1} = \sum_{l=0}^{\min(n,d-1)} (-1)^l \begin{pmatrix} d-1 \\ l \end{pmatrix} \boldsymbol{Q}_n^d(H_{n-l}^d) \boldsymbol{\Sigma}_{n-l}^d \boldsymbol{P}_n^d(S_{n-l}^d).$$

In particular, this yields the norm estimate

$$\|(\boldsymbol{F}_n^d)^{-1}\| \leq 2^{-\frac{n}{2}} \sum_{l=0}^{\min(n,d-1)} \begin{pmatrix} d-1 \\ l \end{pmatrix} \begin{pmatrix} n-l+d-1 \\ d-1 \end{pmatrix} 2^{\frac{l}{2}}.$$

Proof. We define the operator $\sigma_n^d : C(\mathbb{T}^d) \to \Pi_n^{\mathrm{hc}}$,

$$\sigma_n^d = \sum_{\substack{\boldsymbol{j} \in \mathbb{N}_0^d \\ \|\boldsymbol{j}\|_1 = n}} \mathcal{I}_{\boldsymbol{j}},$$

which fulfils the recursion

$$\sigma_n^d = \sum_{l=0}^{n} \sum_{\substack{\boldsymbol{j} \in \mathbb{N}_0^{d-1} \\ \|\boldsymbol{j}\|_1 = l}} \mathcal{I}_{n-l} \otimes \mathcal{I}_{\boldsymbol{j}} = \sum_{l=0}^{n} \mathcal{I}_{n-l} \otimes \sigma_l^{d-1}.$$

Moreover, the interpolation operator obeys $\mathcal{L}_n^1 = \mathcal{I}_n = \sigma_n^1$ and the relation

$$\mathcal{L}_n^d = \sum_{l=0}^{\min(n,d-1)} (-1)^l \begin{pmatrix} d-1 \\ l \end{pmatrix} \sigma_{n-l}^d$$

which is proven by induction over $d \in \mathbb{N}$ using (2.4) in

$$\begin{aligned}
\mathcal{L}_n^d &= \sum_{j=0}^{n} \mathcal{I}_{n-j} \otimes \mathcal{L}_j^{d-1} - \sum_{j=0}^{n-1} \mathcal{I}_{n-1-j} \otimes \mathcal{L}_j^{d-1} \\
&= \sum_{l=0}^{\min(n,d-2)} \sum_{j=l}^{n} (-1)^l \begin{pmatrix} d-2 \\ l \end{pmatrix} \mathcal{I}_{n-j} \otimes \sigma_{j-l}^{d-1} \\
&\quad - \sum_{l=0}^{\min(n-1,d-2)} \sum_{j=l}^{n-1} (-1)^l \begin{pmatrix} d-2 \\ l \end{pmatrix} \mathcal{I}_{n-1-j} \otimes \sigma_{j-l}^{d-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\min(n,d-2)} (-1)^l \binom{d-2}{l} \sigma_{n-l}^d - \sum_{l=1}^{\min(n,d-1)} (-1)^{l-1} \binom{d-2}{l-1} \sigma_{n-l}^d \\
&= \begin{cases} \sum_{l=0}^{n} (-1)^l \binom{d-1}{l} \sigma_{n-l}^d, & n \le d-2 \\ \sum_{l=0}^{d-2} (-1)^l \binom{d-1}{l} \sigma_{n-l}^d + (-1)^{d-1} \sigma_{n-d+1}^d, & n \ge d-1 \end{cases} \\
&= \sum_{l=0}^{\min(n,d-1)} (-1)^l \binom{d-1}{l} \sigma_{n-l}^d.
\end{aligned}
$$

Since $\boldsymbol{\Sigma}_n^d$ and $(\boldsymbol{F}_n^d)^{-1}$ are the matrix representations of the operators σ_n^d and $\mathcal{L}_n^d$, respectively, the assertion follows. The norm estimate finally is due to

$$
\|(\boldsymbol{F}_n^d)^{-1}\|_2 \le \sum_{l=0}^{\min(n,d-1)} \binom{d-1}{l} \left\| \boldsymbol{\Sigma}_{n-l}^d \right\|_2
$$

and

$$
\|\boldsymbol{\Sigma}_n^d\|_2 \le \sum_{\substack{\boldsymbol{j} \in \mathbb{N}_0^d \\ \|\boldsymbol{j}\|_1 = n}} \|\boldsymbol{F}_{\boldsymbol{j}}^{-1}\|_2 = 2^{-\frac{n}{2}} \binom{n+d-1}{d-1}.
$$

■

2.3 A Lagrange interpolant

We proceed by applying the decomposition from Lemma 2.5 to a particular vector of samples in order to get a lower bound for the norm of the inverse Fourier matrix. The specific samples belong to the Lagrange interpolant, interpolating one at the origin and zero at all other sparse grid nodes. We have the following estimates on its Fourier coefficients.

Lemma 2.6. *Let $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $\boldsymbol{e} \in \mathbb{R}^{|S_n^d|}$,*

$$
e_{\boldsymbol{x}} = \begin{cases} 1, & \text{for } \boldsymbol{x} = \boldsymbol{0}, \\ 0, & \text{for } \boldsymbol{x} \in S_n^d \setminus \{\boldsymbol{0}\}, \end{cases}
$$

denote the first unit vector. The Fourier coefficient vector $\hat{\boldsymbol{e}} = (\boldsymbol{F}_n^d)^{-1} \boldsymbol{e} \in \mathbb{C}^{|H_n^d|}$ fulfils

$$
\hat{e}_{\boldsymbol{k}} = 2^{-n} \sum_{l=0}^{\min(n-\ell(\boldsymbol{k}),d-1)} (-2)^l \binom{d-1}{l} \binom{n-l-\ell(\boldsymbol{k})+d-1}{d-1},
$$

see also Lemma 2.2. In particular, we have

i) $\hat{e}_{\boldsymbol{k}} = 2^{-n}$ *for* $\boldsymbol{k} \in H_n^d \setminus H_{n-1}^d$,

ii) $\hat{e}_{\boldsymbol{k}} = \frac{\ell(\boldsymbol{k})-n+1}{2^n}$ *for* $d = 2$, *and*

iii) $|\hat{e}_{\boldsymbol{0}}| \ge \left| \frac{d^n}{n! 2^n} - \mathcal{O}(d^{n-1}) \right|$ *for fixed* $n \in \mathbb{N}_0$ *and* $d > n$.

Proof. The individual summands in the decomposition of Lemma 2.5 can be computed for $l = 0, \dots, \min(n, d-1)$, $\boldsymbol{j} \in \mathbb{N}_0^d$, $\|\boldsymbol{j}\|_1 = n - l$, and $\boldsymbol{k} \in \hat{G}_{\boldsymbol{j}}$ explicitly as

$$(\boldsymbol{F}_{\boldsymbol{j}}^{-1} \boldsymbol{P}_{n-l}^d(G_{\boldsymbol{j}}) \boldsymbol{P}_n^d(S_{n-l}^d)\boldsymbol{e})_{\boldsymbol{k}} = 2^{l-n}.$$

Denoting by $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^{2^{n-l}}$ the vector of all ones, this yields

$$\begin{aligned}
((\boldsymbol{F}_n^d)^{-1}\boldsymbol{e})_{\boldsymbol{k}} &= \left(\sum_{l=0}^{\min(n,d-1)} (-1)^l \binom{d-1}{l} \boldsymbol{Q}_n^d(H_{n-l}^d) \sum_{\substack{\boldsymbol{j} \in \mathbb{N}_0^d \\ \|\boldsymbol{j}\|_1 = n-l}} \boldsymbol{Q}_{n-l}^d(\hat{G}_{\boldsymbol{j}}) 2^{l-n} \mathbf{1} \right)_{\boldsymbol{k}} \\
&= 2^{-n} \sum_{l=0}^{\min(n,d-1)} (-2)^l \binom{d-1}{l} \left| \{\boldsymbol{j} \in \mathbb{N}_0^d : \|\boldsymbol{j}\|_1 = n-l \text{ and } \boldsymbol{k} \in \hat{G}_{\boldsymbol{j}}\} \right| \\
&= 2^{-n} \sum_{l=0}^{\min(n-\ell(\boldsymbol{k}),d-1)} (-2)^l \binom{d-1}{l} \binom{n-l-\ell(\boldsymbol{k})+d-1}{d-1}
\end{aligned}$$

where the last equality follows from Lemma 2.2. We evaluate this sum for the special cases i)-iii). Since $\boldsymbol{k} \in H_n^d \setminus H_{n-1}^d$ yields $\ell(\boldsymbol{k}) = n$, the above sum contains only one summand for $l = 0$ and i) follows. The second assertion, i.e. $d = 2$, follows for $n = \ell(\boldsymbol{k})$ from i) and for $n > \ell(\boldsymbol{k})$, the above sum yields

$$\hat{\boldsymbol{e}}_{\boldsymbol{k}} = \frac{n - \ell(\boldsymbol{k}) + 1}{2^n} - \frac{n - \ell(\boldsymbol{k})}{2^{n-1}} = \frac{\ell(\boldsymbol{k}) - n + 1}{2^n}.$$

Finally, we show iii) by

$$\begin{aligned}
|\hat{\boldsymbol{e}}_{\mathbf{0}}| &= \frac{1}{2^n} \left| \sum_{l=0}^{n} (-2)^l \binom{d-1}{l} \binom{n-l+d-1}{d-1} \right| \\
&= \frac{1}{n! 2^n} \left| \sum_{l=0}^{n} \binom{n}{l} (-2)^l (d-l) \cdot \ldots \cdot (d-l+n-1) \right| \\
&= \left| \frac{d^n}{n! 2^n} - \mathcal{O}(d^{n-1}) \right|.
\end{aligned}$$

■

3 Fixed spatial dimension

Our main result on the norms of the Fourier matrix and its inverse are given for fixed spatial dimension in this section, while the discussion of fixed refinement and increasing spatial dimension is postponed to Section 4. After the above preparation, we are ready to prove the following asymptotic estimates.

Theorem 3.1. *Let the spatial dimension $d \in \mathbb{N}$, $d \geq 2$, be fixed. For $n \in \mathbb{N}$, $n \geq d$, the following bounds are valid*

$$\frac{1}{\sqrt{2^{d-1}(d-2)!}} 2^n n^{\frac{d-2}{2}} - \mathcal{O}(2^n n^{\frac{d-3}{2}}) \leq \|\boldsymbol{F}_n^d\|_2 \leq \frac{1}{2^{d-1}(d-1)!} 2^n n^{d-1} + \mathcal{O}(2^n n^{d-2})$$

and

$$\frac{1}{2^{\frac{d}{2}}\sqrt{(d-1)!}} \frac{n^{\frac{d-1}{2}}}{2^{\frac{n}{2}}} - \mathcal{O}\left(2^{-\frac{n}{2}} n^{\frac{d-2}{2}}\right) \leq \|(\boldsymbol{F}_n^d)^{-1}\|_2 \leq \frac{(\sqrt{2}+1)^{d-1}}{(d-1)!} \frac{n^{d-1}}{2^{\frac{n}{2}}} + \mathcal{O}\left(2^{-\frac{n}{2}} n^{d-2}\right).$$

Proof. The upper bound for the norm of the Fourier matrix easily follows from $\|\boldsymbol{F}_n^d\|_2 \leq \|\boldsymbol{F}_n^d\|_{\mathrm{F}} = |H_n^d|$ and the first relation in Lemma 2.1. Moreover, let $\hat{\boldsymbol{f}} \in \mathbb{R}^{|H_n^d|}$,

$$\hat{f}_{k_1,k_2,\ldots,k_d} = \begin{cases} 2^{-\frac{n}{2}} & k_2 = \ldots = k_d = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $\boldsymbol{f} = \boldsymbol{F}_n^d \hat{\boldsymbol{f}}$ be given, then $\|\hat{\boldsymbol{f}}\|_2 = 1$ and

$$f_{x_1,\ldots,x_d} = 2^{-\frac{n}{2}} \sum_{k_1=-2^{n-1}+1}^{2^{n-1}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}} = 2^{-\frac{n}{2}} \sum_{k_1=-2^{n-1}+1}^{2^{n-1}} \mathrm{e}^{2\pi \mathrm{i} k_1 x_1} = \begin{cases} 2^{\frac{n}{2}} & x_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the lower bound for the norm of the Fourier matrix is due to $\|\boldsymbol{F}_n^d\|_2^2 \geq \|\boldsymbol{f}\|_2^2 = 2^n |S_n^{d-1}|$.

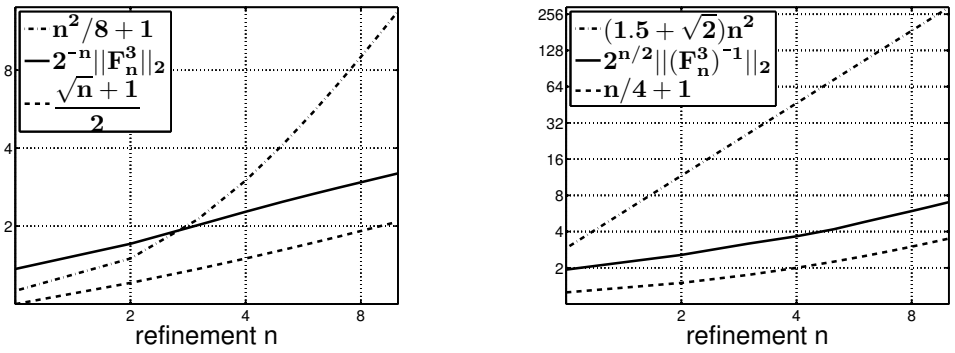
The upper estimate for the inverse Fourier matrix is due to Lemma 2.3 and an induction argument over $d \in \mathbb{N}$. For $d = 1$ and all $n \in \mathbb{N}_0$ we have $\|(\boldsymbol{F}_n)^{-1}\|_2 = 2^{-\frac{n}{2}}$ and using the Boolean sum decomposition, we proceed inductively by

$$\begin{aligned} \|(\boldsymbol{F}_n^d)^{-1}\|_2 &\leq \sum_{j=0}^{n} \|(\boldsymbol{F}_{n-j})^{-1}\|_2 \|(\boldsymbol{F}_j^{d-1})^{-1}\|_2 + \sum_{j=0}^{n-1} \|(\boldsymbol{F}_{n-1-j})^{-1}\|_2 \|(\boldsymbol{F}_j^{d-1})^{-1}\|_2 \\ &\leq \frac{(\sqrt{2}+1)^{d-2}}{(d-2)! 2^{\frac{n}{2}}} \left(\sum_{j=0}^{n} (j+d-2)^{d-2} + \sqrt{2} \sum_{j=0}^{n-1} (j+d-2)^{d-2} \right) \\ &\leq \frac{(\sqrt{2}+1)^{d-1}}{(d-2)! 2^{\frac{n}{2}}} \int_0^{n+1} (j+d-2)^{d-2} \mathrm{d}j \\ &\leq \frac{(\sqrt{2}+1)^{d-1} (n+d-1)^{d-1}}{(d-1)! 2^{\frac{n}{2}}}. \end{aligned}$$

Finally, consider the first unit vector $\boldsymbol{e} = (1, 0, \ldots, 0)^\top \in \mathbb{R}^{|H_n^d|}$ whose Fourier coefficient vector $\hat{\boldsymbol{e}} = (\boldsymbol{F}_n^d)^{-1} \boldsymbol{e}$ fulfils

$$\|(\boldsymbol{F}_n^d)^{-1}\|_2^2 \geq \|\hat{\boldsymbol{e}}\|_2^2 \geq \sum_{\boldsymbol{k} \in H_n^d \setminus H_{n-1}^d} |\hat{e}_{\boldsymbol{k}}|^2 = \frac{|H_n^d \setminus H_{n-1}^d|}{2^{2n}} \geq \frac{n^{d-1}}{2^n 2^d (d-1)!} - \mathcal{O}(2^{-n} n^{d-2}).$$

due to Lemmata 2.6(i) and 2.1(i). ∎



(a) Lower order term of $\|\boldsymbol{F}_n^3\|_2$

(b) Lower order term of $\|(\boldsymbol{F}_n^3)^{-1}\|_2$

Figure 3.1: Lower order term of the norms and asymptotic bounds for the Fourier matrix and its inverse for spatial dimension $d = 3$ and increasing refinement n, cf. Theorem 3.1.

Corollary 3.2. For fixed spatial dimension $d \in \mathbb{N}$ and increasing $n \in \mathbb{N}$, the condition number of $\boldsymbol{F}_n^d$ scales approximately like the $\sqrt{|H_n^d|}$, more precisely the following bounds are valid

$$\Omega(2^{\frac{n}{2}} n^{\frac{2d-3}{2}}) \leq \operatorname{cond}_2 \boldsymbol{F}_n^d \leq \mathcal{O}(2^{\frac{n}{2}} n^{2d-2}).$$

The growth of the condition number with increasing refinement is illustrated in the following Figure 3.2. Beyond the estimate from Corollary 3.2, the condition number increases at higher rates for refinements that are small compared to the spatial dimension.

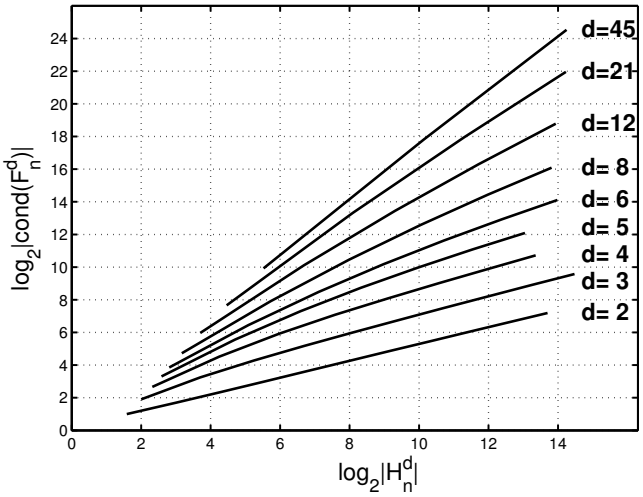


Figure 3.2: Condition number of the Fourier matrix and its inverse for fixed spatial dimension and increasing refinement n. We expect $\log \operatorname{cond}_2 \boldsymbol{F}_n^d \approx \frac{1}{2} \log |H_n^d|$ for large n, cf. Corollary 3.2.

Improvements for $d = 2$

Theorem 3.1 can be refined for the two dimensional case. In particular, we present an identity for the norm of the Fourier matrix and non asymptotic bounds on the norm of the inverse Fourier matrix. However note, that our numerical results in Table 3.1 indicate that the upper bound is not order optimal, see also Figure 3.1(b) for the three dimensional case. We start with the following simple auxiliary result.

Lemma 3.3. *For $n \in \mathbb{N}$ the following identity is fulfilled*

$$\sum_{k=1}^{n}(k+3)2^{k-2}(k+1-n)^2 = 2^n(n-1) + 2 - (n-1)^2.$$

Proof. Summation by parts yields

$$\sum_{k=1}^{n} k2^{k-2} = n2^{n-1} - \frac{1}{2} - \sum_{k=2}^{n} 2^{k-2} = 2^{n-1}(n-1) + \frac{1}{2},$$

and analogously

$$\sum_{k=1}^{n} k^2 2^{k-2} = 2^{n-1}(n^2 - 2n + 3) - \frac{3}{2}, \qquad \sum_{k=1}^{n} k^3 2^{k-2} = 2^{n-1}(n^3 - 3n^2 + 9n - 13) + \frac{13}{2},$$

from which the assertion follows. ∎

Lemma 3.4. *For $n \in \mathbb{N}$ the estimates from Theorem 3.1 can be refined to*

$$\|\boldsymbol{F}_n^2\|_2 = 2^n, \qquad \frac{\sqrt{n-1}}{2^{\frac{n}{2}}} \leq \|(\boldsymbol{F}_n^2)^{-1}\|_2 \leq \frac{(\sqrt{2}+1)n+1}{2^{\frac{n}{2}}}.$$

Proof. Let $\hat{\boldsymbol{f}} \in \mathbb{C}^{|H_n^2|}$, $\|\hat{\boldsymbol{f}}\|_2 = 1$, be arbitrary and set $\hat{\boldsymbol{g}} \in \mathbb{C}^{2^{2n}}$ to

$$\hat{g}_{k_1,k_2} = \begin{cases} \hat{f}_{k_1,k_2} & (k_1,k_2)^\top \in H_n^2, \\ 0 & (k_1,k_2)^\top \in \hat{G}_{n,n} \setminus H_n^2. \end{cases}$$

Moreover, let $\boldsymbol{f} = \boldsymbol{F}_n^2 \hat{\boldsymbol{f}}$ and $\boldsymbol{g} = \boldsymbol{F}_n \otimes \boldsymbol{F}_n \hat{\boldsymbol{g}}$, then

$$\|\boldsymbol{F}_n^2 \hat{\boldsymbol{f}}\|_2^2 = \sum_{\boldsymbol{x} \in S_n^2} |f_{\boldsymbol{x}}|^2 = \sum_{\boldsymbol{x} \in S_n^2} |g_{\boldsymbol{x}}|^2 \leq \sum_{\boldsymbol{x} \in G_{n,n}} |g_{\boldsymbol{x}}|^2 = \|\boldsymbol{F}_n \otimes \boldsymbol{F}_n \hat{\boldsymbol{g}}\|_2^2 = \|\boldsymbol{F}_n \otimes \boldsymbol{F}_n\|_2^2 = 4^n.$$

The three remaining estimates follow along the same lines as in Theorem 3.1. The lower bound for the Fourier matrix is due to $\|\boldsymbol{F}_n^2\|_2^2 \geq 2^n |S_n^1| = 4^n$. The upper estimate for the inverse Fourier matrix follows from

$$\|(\boldsymbol{F}_n^2)^{-1}\|_2 \leq \sum_{j=0}^{n} \|\boldsymbol{F}_j^{-1}\|_2 \|\boldsymbol{F}_{n-j}^{-1}\|_2 + \sum_{j=0}^{n-1} \|\boldsymbol{F}_j^{-1}\|_2 \|\boldsymbol{F}_{n-1-j}^{-1}\|_2 = \frac{n+1}{2^{\frac{n}{2}}} + \frac{n}{2^{\frac{n-1}{2}}}.$$

Finally, the lower bound on the inverse Fourier matrix uses again the first unit vector $\boldsymbol{e} = (1, 0, \ldots, 0)^\top \in \mathbb{R}^{|H_n^2|}$ which yields in conjunction with Lemmata 2.6(ii), 2.1(ii), and 3.3

$$\left\|(\boldsymbol{F}_n^2)^{-1}\right\|_2^2 \geq \sum_{l=0}^{n} \sum_{\boldsymbol{k} \in H_l^2 \setminus H_{l-1}^2} |\hat{e}_{\boldsymbol{k}}|^2 = \frac{(n-1)^2 + \sum_{l=1}^{n}(l+3)2^{l-2}(l-n+1)^2}{2^{2n}} = \frac{2^n(n-1)+2}{2^{2n}}.$$

from which the last assertion easily follows. ∎

n	4	5	6	7	8	9	10	11	12
$\|(\boldsymbol{F}_n^2)^{-1}\|_2$	0.500	0.388	0.298	0.226	0.171	0.128	0.096	0.071	0.053
$\frac{\sqrt{n-1}}{2^{\frac{n}{2}}}$	0.433	0.354	0.280	0.217	0.165	0.125	0.094	0.070	0.052
$\frac{(\sqrt{2}+1)n+1}{2^{\frac{n}{2}}}$	2.664	2.311	1.935	1.582	1.270	1.004	0.786	0.609	0.468

Table 3.1: Matrix norms of $(\boldsymbol{F}_n^2)^{-1}$ and their bounds from Lemma 3.4.

Remark 3.5. Note that the upper bound on the Fourier matrix is improved by an order of magnitude for $d = 2$, but the applied technique gives the suboptimal estimates $\|\boldsymbol{F}_n^d\|_2 \leq \|\bigotimes_{l=1}^d \boldsymbol{F}_n\|_2 = 2^{\frac{nd}{2}}$ for $d > 2$. □

4 Fixed refinement

The second main result on the norms of the Fourier matrix and its inverse are given for fixed refinement and increasing dimension.

Theorem 4.1. *Let the refinement $n \in \mathbb{N}$ be fixed. For spatial dimension $d \in \mathbb{N}$, $d \geq n$, the following bounds are valid*

$$\frac{d^n}{\sqrt{2}n!} - \mathcal{O}(d^{n-\frac{1}{2}}) \leq \|\boldsymbol{F}_n^d\|_2 \leq \frac{d^n}{n!} + \mathcal{O}(d^{n-1})$$

and

$$\frac{d^n}{n!2^n} - \mathcal{O}(d^{n-1}) \leq \|(\boldsymbol{F}_n^d)^{-1}\|_2 \leq \frac{\left(2+\sqrt{2}\right)^n d^n}{n!2^n} + \mathcal{O}(d^{n-1}).$$

Proof. The upper bound on the Fourier matrix can be shown as in Theorem 3.1, i.e., $\|\boldsymbol{F}_n^d\|_2 \leq \|\boldsymbol{F}_n^d\|_{\mathrm{F}} = |H_n^d|$ and by using the first relation in Lemma 2.1. Now, let $\hat{\boldsymbol{f}} \in \mathbb{R}^{|H_n^d|}$, $\hat{f}_{\boldsymbol{k}} = 1$, be given and set $\boldsymbol{f} = \boldsymbol{F}_n^d \hat{\boldsymbol{f}}$. Using the partition (2.3) of the hyperbolic cross and the shorthand notations

$$\begin{aligned}
\boldsymbol{k}^\top &= (k_1 \boldsymbol{k}_1^\top) = (\boldsymbol{k}_0^\top \boldsymbol{k}_n^\top),\ k_1 \in \mathbb{Z},\ \boldsymbol{k}_1 \in \mathbb{Z}^{d-1},\ \boldsymbol{k}_0 \in \mathbb{Z}^n,\ \boldsymbol{k}_n \in \mathbb{Z}^{d-n},\\
\boldsymbol{x}^\top &= (x_1 \boldsymbol{x}_1^\top) = (\boldsymbol{x}_0^\top \boldsymbol{x}_n^\top),\ x_1 \in \mathbb{T},\ \boldsymbol{x}_1 \in \mathbb{T}^{d-1},\ \boldsymbol{x}_0 \in \mathbb{T}^n,\ \boldsymbol{x}_n \in \mathbb{T}^{d-n},\\
\tilde{G}_{\boldsymbol{j}} &= \left(\hat{G}_{j_1} \setminus \hat{G}_{j_1-1}\right) \times \ldots \times \left(\hat{G}_{j_n} \setminus \hat{G}_{j_n-1}\right) \subset \mathbb{Z}^n,\ \boldsymbol{j} \in \mathbb{N}_0^n,
\end{aligned}$$

we obtain

$$\begin{aligned}
f_{\boldsymbol{x}} = \sum_{\boldsymbol{k} \in H_n^d} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}} &= \sum_{j_1=0}^{n} \sum_{k_1 \in \hat{G}_{j_1} \setminus \hat{G}_{j_1-1}} \mathrm{e}^{2\pi \mathrm{i} k_1 x_1} \sum_{\boldsymbol{k}_1 \in H_{n-j_1}^{d-1}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}_1 \boldsymbol{x}_1}\\
&= \sum_{l=0}^{n} \sum_{\substack{\boldsymbol{j} \in \mathbb{N}_0^n \\ \|\boldsymbol{j}\|_1 = l}} \sum_{\boldsymbol{k}_0 \in \tilde{G}_{\boldsymbol{j}}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}_0 \boldsymbol{x}_0} \sum_{\boldsymbol{k}_n \in H_{n-l}^{d-n}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}_n \boldsymbol{x}_n}.
\end{aligned}$$

Since $\boldsymbol{x} \in S_n^d$ implies $\boldsymbol{x} \in G_{\boldsymbol{j}}$ with $\boldsymbol{j} \in \mathbb{N}_0^d$ and $\|\boldsymbol{j}\|_1 = n$, at most n components of $\boldsymbol{j}$ are nonzero. Hence, at most n components of $\boldsymbol{x} \in S_n^d$ are nonzero and without loss of generality let $\boldsymbol{x}_n = (x_{n+1} \dots x_d)^\top = \boldsymbol{0}$. In conjunction with the estimates

$$\sum_{\boldsymbol{k}_0 \in \tilde{G}_{\boldsymbol{0}}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}_0 \boldsymbol{x}_0} = 1, \qquad \left| \sum_{\substack{\boldsymbol{j} \in \mathbb{N}_0^n \\ \|\boldsymbol{j}\|_1 = l}} \sum_{\boldsymbol{k}_0 \in \tilde{G}_{\boldsymbol{j}}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}_0 \boldsymbol{x}_0} \right| \leq C_n, \ 1 \leq l \leq n, \ \boldsymbol{x}_0 \in \mathbb{T}^n,$$

with some constant C_n independent of d, the sample values finally obey

$$|f_{\boldsymbol{x}}| \geq |H_n^{d-n}| - C_n \sum_{l=1}^{n} |H_{n-l}^{d-n}| = \frac{d^n}{n!} - \mathcal{O}(d^{n-1})$$

for all $\boldsymbol{x} \in S_n^d$. Thus $\|\boldsymbol{f}\|_2^2 \geq (d^n/n!)^3 - \mathcal{O}(d^{3n-1})$ and the lower bound on the Fourier matrix follows from $\|\boldsymbol{F}_n^d\|_2 \geq \|\boldsymbol{f}\|_2 / \|\hat{\boldsymbol{f}}\|_2$.

Applying Lemma 2.5, yields the following upper bound on the inverse matrix

$$\begin{aligned}
\|(\boldsymbol{F}_n^d)^{-1}\|_2 &\leq \frac{1}{2^{\frac{n}{2}}} \sum_{l=0}^{n} \binom{d-1}{l} \binom{n-l+d-1}{d-1} 2^{\frac{l}{2}} \\
&= \frac{1}{n! 2^{\frac{n}{2}}} \sum_{l=0}^{n} \binom{n}{l} (d-l+n-1) \cdots (d-l) 2^{\frac{l}{2}} \\
&\leq \frac{(d+n-1)^n}{n! 2^{\frac{n}{2}}} \sum_{l=0}^{n} \binom{n}{l} 2^{\frac{l}{2}} \\
&= \frac{\left(2+\sqrt{2}\right)^n d^n}{n! 2^n} + \mathcal{O}(d^{n-1}).
\end{aligned}$$

Finally, the lower bound on the inverse Fourier matrix uses again the first unit vector $\boldsymbol{e} = (1, 0, \dots, 0)^\top \in \mathbb{R}^{|H_n^d|}$ whose zeroth Fourier coefficient $\hat{e}_{\boldsymbol{0}}$, $\hat{\boldsymbol{e}} = (\boldsymbol{F}_n^d)^{-1} \boldsymbol{e}$, yields

$$\|(\boldsymbol{F}_n^d)^{-1}\|_2 \geq \|\hat{\boldsymbol{e}}\|_2 \geq \hat{e}_{\boldsymbol{0}} \geq \frac{d^n}{n! 2^n} - \mathcal{O}(d^{n-1})$$

due to Lemma 2.6(iii). ∎

Corollary 4.2. For fixed refinement $n \in \mathbb{N}$ and increasing spatial dimension $d \in \mathbb{N}$, the condition number of $\boldsymbol{F}_n^d$ scales approximately like the $|H_n^d|^2$, more precisely the following identity is valid

$$\mathrm{cond}_2 \boldsymbol{F}_n^d = \Theta(d^{2n}).$$

The growth of the condition number with increasing refinement is illustrated in the following Figure 4.1. Beyond the estimate from Corollary 4.2, the condition number increases at slightly lower rates for spatial dimensions that are small compared to the refinement.

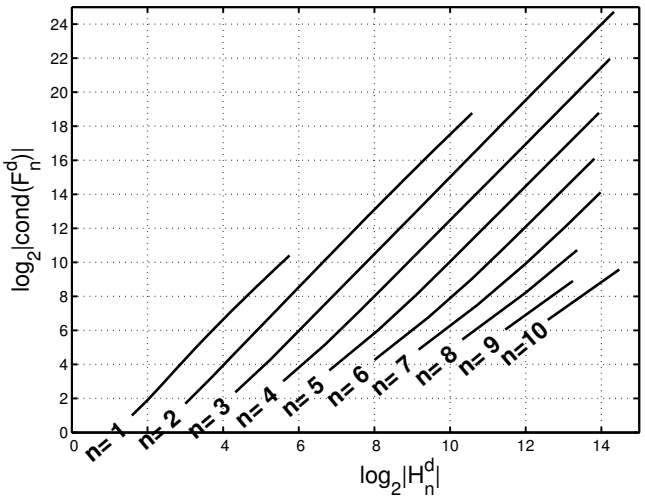


Figure 4.1: Condition number of the Fourier matrix and its inverse for fixed refinement and increasing spatial dimension d. We expect $\log \operatorname{cond}_2 \boldsymbol{F}_n^d \approx 2 \log |H_n^d|$ for large d, cf. Corollary 4.2.

Improvements for $n = 1$

Theorem 4.1 can be refined for the case $n = 1$. In particular, we give a representation of the Fourier matrix and its inverse as rank-2 perturbation of a multiple of the identity matrix. This gives precise information on the eigenvalues and thus on the norms of the matrices in Lemma 4.3. We start with the following observations for the hyperbolic cross and the sparse grid.

$$H_1^d = \left\{ \boldsymbol{k}_0, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_d \in \mathbb{Z}^d \right\} = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

$S_1^d = \frac{1}{2} H_1^d$, and $|H_1^d| = |S_1^d| = d + 1$. Moreover, for all $\boldsymbol{k} \in H_1^d$ and $\boldsymbol{x} \in S_1^d$ holds

$$\boldsymbol{k}\boldsymbol{x} = \boldsymbol{k}^\top \boldsymbol{x} = \begin{cases} \frac{1}{2} & \text{for } \boldsymbol{k} = 2\boldsymbol{x} \neq 0 \\ 0 & \text{otherwise;} \end{cases} \qquad \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}} = \begin{cases} -1 & \text{for } \boldsymbol{k} = 2\boldsymbol{x} \neq 0 \\ 1 & \text{otherwise;} \end{cases}$$

and thus the Fourier matrix is given by

$$\boldsymbol{F}_1^d = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \ldots & 1 & -1 \end{pmatrix} = -2\boldsymbol{I}_{d+1} + \boldsymbol{U}\boldsymbol{U}^\top, \qquad \boldsymbol{U} = \begin{pmatrix} \sqrt{2} & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

Regarding the inverse HCFFT and the computational complexity of this Fourier transform, we obtain

Lemma 4.3. *For fixed refinement $n = 1$ and spatial dimension $d \in \mathbb{N}$, the inverse Fourier matrix $(\boldsymbol{F}_1^d)^{-1} \in \mathbb{R}^{(d+1)\times(d+1)}$ allows for the decomposition*

$$(\boldsymbol{F}_1^d)^{-1} = -\frac{1}{2}\boldsymbol{I}_{d+1} - \boldsymbol{V}\boldsymbol{W}^\top, \qquad \boldsymbol{V} = \begin{pmatrix} -\frac{d-3}{2} & -\frac{1}{2} \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{pmatrix}, \; \boldsymbol{W} = \begin{pmatrix} -\frac{1}{2} & -\frac{d-3}{2} \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}.$$

Moreover, the matrix vector multiplication with $\boldsymbol{F}_1^d$ and its inverse take at most $3d + \mathcal{O}(1)$ floating point operations.

Proof. Due to

$$\boldsymbol{U}^\top\boldsymbol{U} = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & d+1 \end{pmatrix}, \qquad (\boldsymbol{I}_2 - \frac{1}{2}\boldsymbol{U}^\top\boldsymbol{U})^{-1} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{d-1}{2} \end{pmatrix}^{-1} = \begin{pmatrix} d-1 & -\sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix},$$

the Sherman Morrison Woodbury formula yields

$$(\boldsymbol{F}_1^d)^{-1} = -\frac{1}{2}\boldsymbol{I}_{d+1} - \frac{1}{4}\boldsymbol{U}\begin{pmatrix} d-1 & -\sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}\boldsymbol{U}^\top = -\frac{1}{2}\boldsymbol{I}_{d+1} - \frac{1}{4}\begin{pmatrix} 2(d-3) & -2 & \dots & -2 \\ -2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -2 & 0 & \dots & 0 \end{pmatrix}$$

and thus the assertion. The complexity estimate easily follows, since the multiplication with $\boldsymbol{U}^\top$ or $\boldsymbol{W}^\top$ takes $d + \mathcal{O}(1)$, the multiplication with $\boldsymbol{U}$ or $\boldsymbol{V}$ takes $\mathcal{O}(1)$, and the addition with the scaled input vector takes $2d + \mathcal{O}(1)$ floating point operations. ∎

Lemma 4.4. *For $d \in \mathbb{N}$, $d \geq 2$, the estimates from Theorem 4.1 can be refined to*

$$d - 1 \leq \|\boldsymbol{F}_1^d\|_2 \leq d, \qquad \frac{d-1}{2} \leq \|(\boldsymbol{F}_1^d)^{-1}\|_2 \leq \frac{d}{2}.$$

Proof. For $n = 1$ the Fourier matrix and its inverse are symmetric and thus it suffices to compute their extremal eigenvalues. Due to the decomposition (4.1), we have

$$\|\boldsymbol{F}_1^d\|_2 = -2 + \lambda_{\max}(\boldsymbol{U}^\top\boldsymbol{U}) = -2 + \frac{d+3}{2} + \sqrt{\left(\frac{d+3}{2}\right)^2 - 2d}$$

and in conjunction with Lemma 4.3 also

$$\|(\boldsymbol{F}_1^d)^{-1}\|_2 = \frac{1}{2} + \lambda_{\max}(\boldsymbol{V}^\top\boldsymbol{W}) = \frac{1}{2} + \frac{d-3}{4} + \sqrt{\left(\frac{d-3}{4}\right)^2 + \frac{d}{4}}$$

from which the assertions easily follow. ∎

5 Summary

We have shown that the condition number of the hyperbolic cross discrete Fourier transform scales approximately like the square root of the total problem size for fixed spatial dimension and like the square of the total problem size for fixed refinement. In particular, this limits the stability of the hyperbolic cross fast Fourier transform such that a significant loss in accuracy sets in already for moderate spatial dimensions and refinements. Besides standard techniques, we used a Boolean sum decomposition of the associated inverse Fourier matrix which might be of independent interest for the numerical analysis of other sparse grid decompositions.

Acknowledgement

The authors thank the referees for their valuable suggestions and gratefully acknowledge support by the German Research Foundation within the project KU 2557/1-1 and by the Helmholtz Association within the young investigator group VH-NG-526.

References

[1] G. Baszenski and F.-J. Delvos. A discrete Fourier transform scheme for Boolean sums of trigonometric operators. In C. K. Chui, W. Schempp, and K. Zeller, editors, *Multivariate Approximation Theory IV*, ISNM 90, pages 15 – 24. Birkhäuser, Basel, 1989.

[2] H.-J. Bungartz and M. Griebel. Sparse grids. *Acta Numer.*, 13:147 – 269, 2004.

[3] F.-J. Delvos and W. Schempp. *Boolean methods in interpolation and approximation.* Longman Scientific & Technical, Harley, 1989.

[4] M. Döhler, L. Kämmerer, S. Kunis, and D. Potts. NHCFFT, Matlab toolbox for the nonequispaced hyperbolic cross FFT. `http://www.tu-chemnitz.de/~lkae/nhcfft/nhcfft.php`, 2009.

[5] M. Döhler, S. Kunis, and D. Potts. Nonequispaced hyperbolic cross fast Fourier transform. *SIAM J. Numer. Anal.*, 47:4415 – 4428, 2010.

[6] J. Garcke, M. Griebel, and M. Thess. Data mining with sparse grids. *Computing*, 67:225–253, 2001.

[7] V. Gradinaru. Fourier transform on sparse grids: Code design and the time dependent Schrödinger equation. *Computing*, 80:1 – 22, 2007.

[8] M. Griebel, M. Schneider, and C. Zenger. A combination technique for the solution of sparse grid problems. In *Iterative methods in linear algebra (Brussels, 1991)*, pages 263–281. North-Holland, Amsterdam, 1992.

[9] K. Hallatschek. Fouriertransformation auf dünnen Gittern mit hierarchischen Basen. *Numer. Math.*, 63:83 – 97, 1992.

[10] W. Sickel and F. Sprengel. Interpolation on sparse grids and tensor products of Nikol'skij-Besov spaces. *J. Comput. Anal. Appl.*, 1:263 – 288, 1999.

[11] W. Sickel and F. Sprengel. Some error estimates for periodic interpolation of functions from besov spaces. In W. Haussman, K. Jetter, and M. Reimer, editors, *Advances in multivariate approximation*, pages 269 – 287. Wiley-VCH, 1999.

[12] W. Sickel and T. Ullrich. The Smolyak algorithm, sampling on sparse grids and function spaces of dominating mixed smoothness. *East J. Approx.*, 13:387 – 425, 2007.

[13] F. Sprengel. A class of function spaces and interpolation on sparse grids. *Numer. Funct. Anal. Optim.*, 21:273 – 293, 2000.

[14] V. N. Temlyakov. Approximation of functions with bounded mixed derivative. *Proc. Steklov Inst. Math.*, pages vi+121, 1989. A translation of Trudy Mat. Inst. Steklov 178 (1986).

[15] T. Ullrich. Smolyak's algorithm, sampling on sparse grids and Sobolev spaces of dominating mixed smoothness. *East J. Approx.*, 14:1 – 38, 2008.

[16] C. Zenger. Sparse grids. In *Parallel algorithms for partial differential equations (Kiel, 1990)*, volume 31 of *Notes Numer. Fluid Mech.*, pages 241 – 251. Vieweg, Braunschweig, Germany, 1991.